Math 210B Lecture 17 Notes

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1 Localization of Modules, Torsion, Rank, and Local Rings

1.1 Localization of modules

Let R be a commutative ring and $S \subseteq R$ be multiplicatively closed. If M is an R-module, we can define the localization $S^{-1}M$, which is an $S^{-1}R$-module.

Example 1.1. Let S be the set of nonzero non-zero divisors in R. Then $S^{-1}R = Q(R)$ is called the **total quotient ring** of R. The module $S^{-1}M$ is a $Q(R)$-module. If R is an integral domain, Q is a field, so $S^{-1}M$ is a vector space.

If M is and R-module and N is an $S^{-1}R$-module,

$$\operatorname{Hom}_{S^{-1}R}(S^{-1}M, N) \cong \operatorname{Hom}_R(M, N).$$

That is, localization is a left-adjoint to the forgetful functor.

Localization satisfies a universal property: For any $\phi : M \to N$, where N is an $S^{-1}R$-module,

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d] & N \\ S^{-1}M \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

where $\Phi(m/s) = s^{-1}\phi(m)$.

Proposition 1.1. *$S^{-1}M \cong S^{-1}R \otimes_R M$ as $S^{-1}R$-modules.*

Proof. Let $S^{-1}R \times M \to S^{-1}M$ send $(r/s, m) \mapsto (rm)/s$. This is left $S^{-1}R$-linear and right R-linear, so we get a map $S^{-1}R \otimes RM \to S^{-1}M$ of $S^{-1}R$-modules. Conversely, we have the R-module homomorphism $M \to S^{-1}R \otimes_R M$ sending $m \mapsto 1 \otimes m$. The universal property gives a map $S^{-1}M \to S^{-1}R \otimes_R M$ sending $m/s \mapsto s^{-1} \otimes m$. Check that these are inverse maps. $\square$

1.2 Torsion and rank

Let $Q = Q(R)$ be the total quotient ring of R.

Definition 1.1. If M is an R-module, then $m \in M$ is **torsion** if there exists some $r \in S$ such that $rm = 0$.

$M_{\text{tor}} = \{m \in M : m \text{ torsion}\}$ is an R-submodule of M.

Lemma 1.1. $M_{\text{tor}} = \ker(M \to Q \otimes_R M)$.

Proof. $m \in M_{\text{tor}}$ iff $m/1 = 0$ in $Q \otimes_R M$, since this is isomorphic to $S^{-1}M$. □

Example 1.2. Let $A = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Then $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} = A_{\text{tor}}$ is the torsion part.

Definition 1.2. We say M is **torsion-free** if $M_{\text{tor}} = 0$.

Definition 1.3. The **annihilator** of M (in R) is $\text{Ann}(M) := \{r \in R : rm = 0 \, \forall m \in M\}$.

This is an ideal of R.

Lemma 1.2. *If R is an integral domain and M is finitely generated over R, then $\text{Ann}(M) \neq 0$ if and only if $M = M_{\text{tor}}$.*

Proof. ($\implies$): If $\text{Ann}(M) \neq 0$, then there exists some $r \neq 0$ in M such taht $rm = 0$ for all $m \in M$. So $m \in M_{\text{tor}}$ for all $m \in M$.

($\impliedby$): Let $m_1, \ldots, m_n \in M$ generated M as an R-module. Let $e_1, \ldots, r_n \in R \setminus \{0\}$ be such that $r_i m_i = 0$ for all i. THen $r_1 \cdots r_n m = 0$ for all $m \in M$. Since R is an integral domain, $r_1 \cdots r_n \neq 0$, so $r_1 \cdots r_n \in \text{Ann}(M)$. □

Definition 1.4. The **rank** of an R-module over an integral domain R is $\text{rank}_R(M) = \dim_Q(Q \otimes_R M)$, if this dimension is finite.

Proposition 1.2. $\text{rank}_R(M)$ *is the maximal number of R-linearly independent elements in M.*

Proof. An element of M_{tor} is by itself linearly dependent. We may replace M by M/M_{tor}, so we may suppose M is R-torsion free. Them $M \to Q \otimes_R M$ is an injection. M has $\leq \dim_Q(Q \otimes_R M) = \text{rank}_R(M) =: n$ linearly independent elements. If $v_1, \ldots, v_n \in Q \otimes_R M$ is a basis over Q, then there exists some $r \in R$ such that $rv_1, \ldots, rv_n \in M$, and the rv_i are R-linearly independent. So we have at least n R-linearly independent elements in M. □

1.3 Local rings

Definition 1.5. A commutative ring R is **local** if it has a unique maximal ideal m.

If R is local, R/m is a field, called the **residue field** of R.

Proposition 1.3. *Let R be commutative, and let $p \subseteq R$ be a prime ideal. Then R_p is a local ring with maximal ideal pR_p. The ideals of R_p are R_p and IR_p with $I \subseteq p$.*

Lemma 1.3. *If R is local and m is maximal, then $R \setminus m = R^\times$.*

Proof. If $a \in R \setminus m$, then $(a) = R$. So $a \in R^\times$. Conversely, if $a \notin R^\times$, then $(a) \neq R$, so $(a) \subseteq m$. So $a \in m$. □

Lemma 1.4. *If R is commutative an $m \subseteq R$ is maximal, then $R/m \cong R_m/mR_m$.*

Proof. Look at $R/m \to R_m/mR_m$ given by $r + m \mapsto r/1 + mR_m$. These are both fields, so this is an injection. If $r \in R$ and $u \in R \setminus m$, then there eixsts some $r \in R \setminus m$ such that $uv = 1 \bmod m$. Then $vr + m \mapsto (vr)/1 + mR_m = r/n + mR_m$. So this is onto. □

Proposition 1.4. *Let R be commutative and M be an R-module. The following are equivalent.*

1. $M = 0$
2. $M_p = 0$ *for all prime ideals* $p \subseteq R$
3. $M_m = 0$ *for all maximal ideals* $m \subseteq R$.

Proof. Each of these is a special case of the last, so we just need to show (3) $\implies$ (1). Let $m \in M \setminus \{0\}$. Let $U = \operatorname{Ann}(R_m) = \{r \in M : rm = 0\}$. I is proper, so $I \subseteq m$ for some maximal ideal m.[1] If $r/u \in R_m$ is such that $(r/u)m = 0 \in M_m$, then there exists $s \in R \setminus m$ such that $srm = 0$. Then $sr \in m$, so $r \in m$ as m is prime. So $\operatorname{Ann}(R_m m) \subsetneq R_m$. Then $m/1 \neq 0$ in R_m. □

Next time, we will prove the following important theorem.

Lemma 1.5 (Nakayama)**.** *If M is a finitely generated module over a local ring (R, m) such that $mM = M$, then $M = 0$.*

Remark 1.1. What does the condition $mM = M$ mean? M/mM is an R/m-vector space. This says that if $M/mM = 0$, then $M = 0$.

[1] This uses Zorn's lemma.